

# Casimir densities for a spherical brane in Rindler-like spacetimes

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## Abstract

Wightman function, the vacuum expectation values of the field square and the energy-momentum tensor are evaluated for a scalar field obeying mixed boundary condition on a spherical brane in  $(D + 1)$ -dimensional Rindler-like spacetime  $Ri \times S^{D-1}$ , where  $Ri$  is a two-dimensional Rindler spacetime. This spacetime approximates the near horizon geometry of  $(D + 1)$ -dimensional black hole in the large mass limit. The vacuum expectation values are presented as the sum of boundary-free and brane-induced parts. Further we extract from the Wightman function for the boundary-free geometry the corresponding function in the bulk  $R^2 \times S^{D-1}$ . For the latter geometry the vacuum expectation values of the field square and the energy-momentum tensor do not depend on the spacetime point. For the renormalization of these quantities we use zeta regularization technique. Various limiting cases of the brane-induced vacuum expectation values are investigated.

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## 1 Introduction

Motivated by string/M theory, the AdS/CFT correspondence, and the hierarchy problem of particle physics, braneworld models were studied actively in recent years [1]-[4]. In this models, our universe is realized as a boundary of a higher dimensional spacetime. In particular, a well studied example is when the bulk is an AdS space. In the cosmological context, embedding of a four dimensional Friedmann-Robertson-Walker universe was also considered when the bulk is described by AdS or dS black hole [5, 6]. In the latter case, the mass of the black hole was found to effectively act as an energy density on the brane with the same equation of state of radiation. Representing radiation as conformal matter and exploiting AdS/CFT correspondence, the Cardy-Verlinde formula [7] for the entropy was found for the universe (for the entropy formula in the case of dS black hole see [8]). Moreover, in the AdS/CFT correspondence, the case of a bulk AdS black hole represents a different phase of the same theory and there is the exciting connection that a transition between an ordinary bulk AdS and a bulk AdS black hole corresponds to the confinement-de confinement transition in the dual CFT [9]. Therefore it seems interesting to generalize the study of quantum effects due to bulk AdS black holes.

The investigation of quantum effects in braneworld models is of considerable phenomenological interest, both in particle physics and in cosmology. The braneworld corresponds to a manifold with dynamical boundaries and all fields which propagate in the bulk will give Casimir-type contributions to the vacuum energy (for reviews of the Casimir effect see Refs. [10]), and as a result

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to the vacuum forces acting on the branes. In dependence of the type of a field and boundary conditions imposed, these forces can either stabilize or destabilize the braneworld. In addition, the Casimir energy gives a contribution to both the brane and bulk cosmological constants and, hence, has to be taken into account in the self-consistent formulation of the braneworld dynamics. Motivated by these, the role of quantum effects in braneworld scenarios has received a great deal of attention. For a conformally coupled scalar this effect was initially studied in Ref. [11] in the context of M-theory, and subsequently in Refs. [12, 13] for a background Randall–Sundrum geometry. The models with dS and AdS branes, and higher dimensional brane models are considered as well [14].

In view of these recent developments, it seems interesting to generalize the study of quantum effects to other types of bulk spacetimes. In particular, it is of interest to consider non-Poincaré invariant braneworlds, both to better understand the mechanism of localized gravity and for possible cosmological applications. Bulk geometries generated by higher-dimensional black holes are of special interest. In these models, the tension and the position of the brane are tuned in terms of black hole mass and cosmological constant and brane gravity trapping occurs in just the same way as in the Randall–Sundrum model. Braneworlds in the background of the AdS black hole were studied in [6]. Like pure AdS space the AdS black hole may be superstring vacuum. It is of interest to note that the phase transitions which may be interpreted as confinement-deconfinement transition in AdS/CFT setup may occur between pure AdS and AdS black hole [9]. Though, in the generic black hole background the investigation of brane-induced quantum effects is technically complicated, the exact analytical results can be obtained in the near horizon and large mass limit when the brane is close to the black hole horizon. In this limit the black hole geometry may be approximated by the Rindler-like manifold (for some investigations of quantum effects on background of Rindler-like spacetimes see [15] and references therein). In the present paper we investigate the Wightman function, the vacuum expectation values of the field square and the energy-momentum tensor for a scalar field with an arbitrary curvature coupling parameter for the spherical brane on the bulk  $Ri \times S^{D-1}$ , where  $Ri$  is a two-dimensional Rindler spacetime. Note that the corresponding quantities induced by a single and two parallel flat branes in the bulk geometry  $Ri \times R^{D-1}$  for both scalar and electromagnetic fields are investigated in [16]. This problem is also of separate interest as an example with gravitational and boundary-induced polarizations of the vacuum, where all calculations can be performed in a closed form. The paper is organized as follows. In section 2 we consider the positive frequency Wightman function in the region between the brane and Rindler horizon. This function is presented as the sum of boundary-free and boundary-induced parts. The vacuum expectation values for the boundary-free geometry are investigated in section 3. The vacuum expectation values induced by a spherical brane are studied in section 4. Section 5 summarizes the main results of the paper.

## 2 Wightman function

Let us consider a scalar field  $\varphi(x)$  propagating on background of  $(D + 1)$ -dimensional Rindler-like spacetime  $Ri \times S^{D-1}$ , where  $Ri$  is a two-dimensional Rindler spacetime. The corresponding metric is described by the line element

$$ds^2 = \xi^2 d\tau^2 - d\xi^2 - r_H^2 d\Sigma_{D-1}^2, \quad (1)$$

with the Rindler-like  $(\tau, \xi)$  part and  $d\Sigma_{D-1}^2$  is the line element for the space with positive constant curvature with the Ricci scalar  $R = (D - 2)(D - 1)/r_H^2$ . Line element (1) describes the near horizon geometry of  $(D + 1)$ -dimensional topological black hole with the line element [17]

$$ds^2 = A_H(r) dt^2 - \frac{dr^2}{A_H(r)} - r^2 d\Sigma_{D-1}^2, \quad (2)$$

where

$$A_H(r) = k + \frac{r^2}{l^2} - \frac{r_0^{n+2}}{l^2 r^n}, \quad n = D - 2, \quad (3)$$

and the parameter  $k$  classifies the horizon topology, with  $k = 0, -1, 1$  corresponding to flat, hyperbolic, and elliptic horizons, respectively. In (3) the parameter  $l$  is related to the bulk cosmological constant and the parameter  $r_0$  depends on the mass of the black hole and on the bulk gravitational constant. In the non extremal case the function  $A_H(r)$  has a simple zero at  $r = r_H$ . In the near horizon limit, introducing new coordinates  $\tau$  and  $\rho$  in accordance with

$$\tau = \frac{1}{2} A'_H(r_H) t, \quad r - r_H = \frac{1}{4} A'_H(r_H) \xi^2, \quad (4)$$

the line element is written in the form (1). Note that for a  $(D + 1)$ -dimensional Schwarzschild black hole [18] one has  $A_H(r) = 1 - (r_H/r)^{D-2}$  and, hence,  $A'_H(r_H) = n/r_H$ .

The field equation is in the form

$$\left( g^{ik} \nabla_i \nabla_k + m^2 + \zeta R \right) \varphi(x) = 0, \quad (5)$$

where  $\zeta$  is the curvature coupling parameter. Below we will assume that the field satisfies the Robin boundary condition on the hypersurface  $\xi = a$ :

$$\left( A + B \frac{\partial}{\partial \xi} \right) \varphi = 0, \quad \xi = a, \quad (6)$$

with constant coefficients  $A$  and  $B$ . The Dirichlet and Neumann boundary conditions are obtained as special cases. In accordance with (4), the hypersurface  $\xi = a$  corresponds to the spherical shell near the black hole horizon with the radius  $r_a = r_H + A'_H(r_H)a^2/4$ . Here we consider the general case of the ratio  $A/B$ . The application to the braneworld scenario will be given below.

To evaluate the vacuum expectation values of the field square and the energy-momentum tensor we need a complete set of eigenfunctions satisfying the boundary condition (6). Below we shall use the hyperspherical angular coordinates  $(\vartheta, \phi) = (\theta_1, \theta_2, \dots, \theta_n, \phi)$  on  $S^{D-1}$  with  $0 \leq \theta_k \leq \pi$ ,  $k = 1, \dots, n$ , and  $0 \leq \phi \leq 2\pi$ . In these coordinates the variables are separated and the eigenfunctions can be written in the form

$$\varphi_\alpha(x) = C_\alpha f(\xi) Y(m_k; \vartheta, \phi) e^{-i\omega\tau}, \quad (7)$$

where  $m_k = (m_0 \equiv l, m_1, \dots, m_n)$ , and  $m_1, m_2, \dots, m_n$  are integers such that

$$0 \leq m_{n-1} \leq \dots \leq m_1 \leq l, \quad -m_{n-1} \leq m_n \leq m_{n-1}, \quad (8)$$

$Y(m_k; \vartheta, \phi)$  is the surface harmonic of degree  $l$  [19]. Substituting this into Eq. (5) we see that the function  $f(\xi)$  satisfies the equation

$$\xi \frac{d}{d\xi} \left( \xi \frac{df}{d\xi} \right) + (\omega^2 - \xi^2 \lambda_l^2) f(\xi) = 0, \quad (9)$$

with the notation

$$\lambda_l = \frac{1}{r_H} \sqrt{l(l+n) + \zeta n(n+1) + m^2 r_H^2}. \quad (10)$$

The linearly independent solutions to (9) are the Bessel modified functions  $I_{\pm i\omega}(\lambda_l \xi)$  and  $K_{i\omega}(\lambda_l \xi)$  with the imaginary order.

In the region  $0 < \xi < a$  the solution to (9) satisfying boundary condition (6) has the form

$$f(\xi) = Z_{i\omega}(\lambda_l \xi, \lambda_l a) \equiv K_{i\omega}(\lambda_l \xi) - \frac{\bar{K}_{i\omega}(\lambda_l a)}{\bar{I}_{i\omega}(\lambda_l a)} I_{i\omega}(\lambda_l \xi), \quad (11)$$

where for a given function  $F(z)$  we use the notation

$$\bar{F}(z) = AF(z) + bzF'(z) = 0, \quad b = B/a. \quad (12)$$

The coefficient  $C_\alpha$  in (7) can be found from the normalization condition

$$\int |\varphi_\alpha(x)|^2 \sqrt{-g} dV = \frac{1}{2\omega}, \quad (13)$$

where the integration goes over the region between the horizon and the sphere. Substituting eigenfunctions (7), using the relation

$$\int |Y(m_k; \vartheta, \phi)|^2 d\Omega = N(m_k) \quad (14)$$

for spherical harmonics, one finds

$$C_\alpha = \frac{1}{\pi} \sqrt{\frac{\sinh \omega \pi}{r_H^{n+1} N(m_k)}}. \quad (15)$$

The explicit form for  $N(m_k)$  is given in [19] and will not be necessary for the following considerations in this paper.

First of all we evaluate the positive frequency Wightman function

$$G^+(x, x') = \langle 0 | \varphi(x) \varphi(x') | 0 \rangle, \quad (16)$$

where  $|0\rangle$  is the amplitude for the corresponding vacuum state. By expanding the field operator over eigenfunctions and using the commutation relations one can see that

$$G^+(x, x') = \sum_\alpha \varphi_\alpha(x) \varphi_\alpha^*(x'). \quad (17)$$

Substituting eigenfunctions (7) with the function (11) into this mode sum and by making use the addition theorem

$$\sum_{m_k} \frac{1}{N(m_k)} Y(m_k; \vartheta, \phi) Y(m_k; \vartheta', \phi') = \frac{2l+n}{nS_D} C_l^{n/2}(\cos \theta), \quad (18)$$

for the Wightman function one finds

$$\begin{aligned} G^+(x, x') &= \frac{r_H^{1-D}}{\pi^2 n S_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos \theta) \\ &\times \int_0^\infty d\omega \sinh(\omega \pi) e^{-i\omega(\tau-\tau')} Z_{i\omega}(\lambda_l \xi, \lambda_l a) Z_{i\omega}^*(\lambda_l \xi', \lambda_l a). \end{aligned} \quad (19)$$

In (18),  $S_D = 2\pi^{D/2}/\Gamma(D/2)$  is the total area of the surface of the unit sphere in  $D$ -dimensional space,  $C_l^{n/2}(x)$  is the Gegenbauer or ultraspherical polynomial of degree  $l$  and order  $n/2$ ,  $\theta$  is the angle between directions  $(\vartheta, \phi)$  and  $(\vartheta', \phi')$ , and the sum is taken over the integer values

$m_k$ ,  $k = 1, 2 \dots$  in accordance with (8). To transform the expression on the right of (19), we present the product of the functions  $Z_{i\omega}$  in the form

$$\begin{aligned} Z_{i\omega}(\lambda\xi, \lambda a)Z_{i\omega}^*(\lambda\xi', \lambda a) &= K_{i\omega}(\lambda\xi)K_{i\omega}(\lambda\xi') + \frac{\pi\bar{K}_{i\omega}(\lambda a)}{2i \sinh \pi\omega} \\ &\times \sum_{\sigma=-1,1} \frac{I_{i\sigma\omega}(\lambda\xi)I_{i\sigma\omega}(\lambda\xi')}{\sigma\bar{I}_{i\sigma\omega}(\lambda a)}. \end{aligned} \quad (20)$$

On the base of this formula from (19) one finds

$$\begin{aligned} G^+(x, x') &= G_0^+(x, x') + \frac{r_H^{1-D}}{2i\pi n S_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos\theta) \\ &\times \int_0^\infty d\omega e^{-i\omega(\tau-\tau')} \bar{K}_{i\omega}(\lambda_l a) \sum_{\sigma=-1,1} \frac{I_{i\sigma\omega}(\lambda_l \xi)I_{i\sigma\omega}(\lambda_l \xi')}{\sigma\bar{I}_{i\sigma\omega}(\lambda_l a)}, \end{aligned} \quad (21)$$

where the part

$$\begin{aligned} G_0^+(x, x') &= \frac{r_H^{1-D}}{\pi^2 n S_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos\theta) \\ &\times \int_0^\infty d\omega \sinh(\omega\pi) e^{-i\omega(\tau-\tau')} K_{i\omega}(\lambda_l \xi) K_{i\omega}(\lambda_l \xi') \end{aligned} \quad (22)$$

does not depend on the parameter  $a$  determining the radius of the spherical shell and corresponds to the Wightman function in the situation when the spherical shell is absent. Assuming that the function  $\bar{I}_{i\omega}(\lambda a)$  ( $\bar{I}_{-i\omega}(\lambda a)$ ) has no zeros for  $-\pi/2 \leq \arg\omega < 0$  ( $0 < \arg\omega < \pi/2$ ) we can rotate the integration contour over  $\omega$  by angle  $-\pi/2$  for the term with  $\sigma = 1$  and by angle  $\pi/2$  for the term with  $\sigma = -1$ . The integrals taken around the arcs of large radius tend to zero under the condition  $|\xi\xi'| < a^2 e^{|\tau-\tau'|}$  (note that, in particular, this is the case in the coincidence limit for the region under consideration). As a result for the Wightman function one obtains

$$G^+(x, x') = G_0^+(x, x') + \langle \varphi(x)\varphi(x') \rangle^{(b)}, \quad (23)$$

where for the sphere-induced part one has

$$\begin{aligned} \langle \varphi(x)\varphi(x') \rangle^{(b)} &= -\frac{r_H^{1-D}}{\pi n S_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos\theta) \\ &\times \int_0^\infty d\omega \frac{\bar{K}_\omega(\lambda_l a)}{I_\omega(\lambda_l a)} I_\omega(\lambda_l \xi) I_\omega(\lambda_l \xi') \cosh[\omega(\tau - \tau')]. \end{aligned} \quad (24)$$

For the points away the brane this part is finite in the coincidence limit.

We have investigated the Whightman function in the region between the horizon and the boundary located at  $\xi = a$  for an arbitrary ratio of boundary coefficients  $A/B$ . In the corresponding braneworld scenario the geometry is made up by two slices of the region  $0 < \xi < a$  glued together at the brane with a orbifold-type symmetry condition analogous to that in the Randall-Sundrum model and the ratio  $A/B$  for bulk scalars is related to the brane mass parameter of the field and the extrinsic curvature of the brane. The corresponding formula is obtained by the way similar to that in the case of the Randall-Sundrum braneworld (see, for instance, [13, 20]). For this we note that in braneworlds the action for a scalar field with general curvature coupling parameter, in addition to the bulk action contains a surface action in the form  $\int d^{D-1}x \sqrt{h}(c + \zeta K)\varphi^2$ , where the integration goes over the brane,  $h$  is the absolute value of the

determinant for the corresponding induced metric,  $c$  is the brane mass parameter for the field, and  $K$  is the extrinsic curvature scalar for the brane. This action gives  $\delta$ -type contributions to the field equation located on the brane. Now the eigenfunctions for the quantized bulk scalar field can be written in the form (7), where the function  $f(\xi)$  is a solution to the equation which differs from (9) by the presence of the term  $-(c + \zeta K)\xi^2 f(\xi)\delta(\xi - a)$  on the right hand side. To obtain the boundary condition for the function  $f(\xi)$  we integrate the corresponding equation about  $\xi = a$ . Assuming that the function  $f(\xi)$  is continuous at this point one finds

$$\lim_{\epsilon \rightarrow 0} \frac{df}{d\xi} \Big|_{\xi=a-\epsilon}^{\xi=a+\epsilon} = (c + \zeta K)f(a). \quad (25)$$

For an untwisted scalar field we have  $f(\xi) = f(2a - \xi)$  and from (25) we obtain the boundary condition in the form (6) with

$$\frac{A}{B} = \frac{1}{2} \left( c - \frac{\zeta}{a} \right), \quad (26)$$

where we have taken into account that for the boundary under consideration  $K = -1/a$ . For a twisted scalar  $f(\xi) = -f(2a - \xi)$  and from (25) we obtain the Dirichlet boundary condition. Note that in the braneworld bulk the integration in the normalization integral goes over two copies of the bulk manifold. This leads to the additional coefficient 1/2 in the expression (15) for the normalization coefficient  $C_\alpha$ . Hence, the Wightman function in the orbifolded braneworld case is given by formula (23) with an additional factor 1/2 in formulae (22), (24). As it has been mentioned above this function corresponds to the braneworld in the AdS black hole bulk in the limit when the brane is close to the black hole horizon.

### 3 Boundary-free geometry

In this section we will consider the vacuum expectation values for the geometry without boundaries. First of all we note that the corresponding Wightman function can be presented in the form

$$G_0^+(x, x') = \tilde{G}_0^+(x, x') - \frac{r_H^{1-D}}{\pi^2 n S_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos \theta) \times \int_0^\infty d\omega e^{-\omega\pi} \cos[\omega(\tau - \tau')] K_{i\omega}(\lambda_l \xi) K_{i\omega}(\lambda_l \xi'), \quad (27)$$

with the function

$$\tilde{G}_0^+(x, x') = \frac{r_H^{1-D}}{\pi^2 n S_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos \theta) \times \int_0^\infty d\omega \cosh\{\omega[\pi - i(\tau - \tau')]\} K_{i\omega}(\lambda_l \xi) K_{i\omega}(\lambda_l \xi'). \quad (28)$$

In this formula the  $\omega$ -integral can be evaluated with the result

$$\tilde{G}_0^+(x, x') = \frac{r_H^{1-D}}{2\pi n S_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos \theta) \times K_0\left(\lambda_l \sqrt{\xi^2 + \xi'^2 - 2\xi \xi' \cosh(\tau - \tau')}\right). \quad (29)$$

It can be checked that this function is the Wightman function for the bulk geometry  $R^2 \times S^{D-1}$  described by the line element

$$ds^2 = dt^2 - (dx^1)^2 - r_H^2 d\Sigma_{D-1}^2, \quad (30)$$

where the coordinates  $(t, x^1)$  are related to the coordinates  $(\tau, \xi)$  by formulas  $t = \xi \sinh \tau$ ,  $x^1 = \xi \cosh \tau$ . To see this we note that the normalized eigenfunctions corresponding to this geometry are given by the formula

$$\tilde{\varphi}_\alpha(x) = \frac{Y(m_k; \vartheta, \phi) e^{ik_1 x^1 - i\omega_l t}}{\sqrt{4\pi\omega_l N(m_k)r_H^{n+1}}}, \quad (31)$$

where  $\alpha = (k_1, m_k)$  and  $\omega_l^2 = k_1^2 + \lambda_l^2$ , with  $\lambda_l$  defined by relation (10). Substituting these functions into the corresponding mode sum and evaluating the  $k_1$ -integral, for the case  $|x^1 - x^{1'}| > |t - t'|$  one finds

$$\begin{aligned} \tilde{G}_0^+(x, x') &= \sum_{\alpha} \tilde{\varphi}_{\alpha}(x) \tilde{\varphi}_{\alpha}^*(x') \\ &= \frac{r_H^{1-D}}{2\pi n S_D} \sum_{l=0}^{\infty} (2l+n) C_l^{n/2}(\cos \theta) K_0 \left( \lambda_l \sqrt{(x^1 - x^{1'})^2 - (t - t')^2} \right). \end{aligned} \quad (32)$$

Noting that  $\xi^2 + \xi'^2 - 2\xi\xi' \cosh(\tau - \tau') = (x^1 - x^{1'})^2 - (t - t')^2$  we see that this formula coincides with (29).

In formula (27), the divergences in the coincidence limit are contained in the term  $\tilde{G}_0^+(x, x')$  and, hence, the renormalization is needed for this term only. Now we turn to the evolution of the vacuum expectation values of the field square and the energy-momentum tensor for the geometry  $R^2 \times S^{D-1}$  described by the line element (30). The amplitude of the corresponding vacuum state we will denote by  $|\tilde{0}\rangle$ . First of all note that from the problem symmetry it follows that the expectation values  $\langle \tilde{0} | \varphi^2 | \tilde{0} \rangle$ ,  $\langle \tilde{0} | T_i^i | \tilde{0} \rangle$  do not depend on the point of observation and

$$\begin{aligned} \langle \tilde{0} | T_0^0 | \tilde{0} \rangle &= \langle \tilde{0} | T_1^1 | \tilde{0} \rangle, \\ \langle \tilde{0} | T_2^2 | \tilde{0} \rangle &= \dots = \langle \tilde{0} | T_D^D | \tilde{0} \rangle. \end{aligned} \quad (33)$$

The component  $\langle \tilde{0} | T_2^2 | \tilde{0} \rangle$  can be expressed through the energy density by using the trace relation

$$T_i^i = D(\zeta - \zeta_c) \nabla_i \nabla^i \varphi^2 + m^2 \varphi^2. \quad (34)$$

From this relation it follows that

$$\langle \tilde{0} | T_2^2 | \tilde{0} \rangle = \frac{1}{D-1} (m^2 \langle \tilde{0} | \varphi^2 | \tilde{0} \rangle - 2 \langle \tilde{0} | T_0^0 | \tilde{0} \rangle). \quad (35)$$

Hence, it is sufficient to find the renormalized vacuum expectation values of the field square and the energy density. Using the eigenmodes (31), these quantities are presented as mode sums

$$\langle \tilde{0} | \varphi^2 | \tilde{0} \rangle = \frac{r_H^{-n}}{4\pi S_D} \int_{-\infty}^{+\infty} dk_1 \sum_l \frac{D_l}{\eta_l(r_H k_1)}, \quad (36)$$

$$\langle \tilde{0} | T_0^0 | \tilde{0} \rangle = \frac{r_H^{-n-2}}{4\pi S_D} \int_{-\infty}^{+\infty} dk_1 \sum_l D_l \eta_l(r_H k_1), \quad (37)$$

with the notation  $\eta_l(x) = \sqrt{x^2 + r_H^2 \lambda_l^2}$ . In this formulas

$$D_l = (2l + D - 2) \frac{\Gamma(l + D - 2)}{\Gamma(D - 1) l!} \quad (38)$$

is the degeneracy of each angular mode with given  $l$ . Of course, quantities (36), (37) are divergent and some renormalization procedure is needed. As such a procedure we will use the zeta function technique. Let us define the zeta function

$$\begin{aligned}\zeta(s) &= \int_{-\infty}^{+\infty} dx \sum_{l=0}^{\infty} D_l \eta_l^{-2s}(x) \\ &= \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \zeta_{S^{n+1}} \left( s - \frac{1}{2} \right),\end{aligned}\quad (39)$$

where

$$\begin{aligned}\zeta_{S^{n+1}}(z) &= \sum_{l=0}^{\infty} D_l (r_H \lambda_l)^{-2z} \\ &= \sum_{l=0}^{\infty} D_l [(l + n/2)^2 + b_n]^{-z},\end{aligned}\quad (40)$$

is the zeta function for a scalar field on the spacetime  $R \times S^{n+1}$  and

$$b_n = \zeta n(n+1) - n^2/4 + m^2 r_H^2. \quad (41)$$

This function is well investigated in literature (see, for example, [21]) and can be presented as a series of incomplete zeta functions. Here we recall that the function  $\zeta_{S^{n+1}}(z)$  is a meromorphic function with simple poles at  $z = (n+1)/2 - j$ , where  $j = 0, 1, 2, \dots$  for  $n$  even and  $0 \leq j \leq (n-1)/2$  for  $n$  odd. For  $n$  even one has  $\zeta_{S^{n+1}}(-j) = 0$ ,  $j = 1, 2, \dots$ . Note that the function  $\zeta_{S^{n+1}}(z)$  can be expressed in terms of the function

$$F(z, c, b) = \sum_{l=1}^{\infty} [(l + c)^2 + b]^{-z}, \quad (42)$$

for  $n$  even or its derivative for  $n$  odd as

$$\zeta_{S^{n+1}}(z) = \frac{2^{1-2\eta}}{\Gamma(D-1)} \sum_{j=0}^{[n/2]} \frac{b_j^{(\eta)}(b_n)}{(j-z+2\eta)^{2\eta}} \frac{\partial^{2\eta}}{\partial c^{2\eta}} F(z-j-2\eta, c, b_n) \Big|_{c=\eta}, \quad (43)$$

where  $\eta = n/2 - [n/2]$  and the square brackets mean the integer part of the enclosed expression. In formula (43) the coefficients  $b_j^{(\eta)}(b_n)$  are defined by the relation

$$\prod_{q=1}^{[n/2]} [y - (q + \eta - 1)^2 - b_n] = \sum_{j=0}^{[n/2]} b_j^{(\eta)}(b_n) y^j. \quad (44)$$

The formulas for the analytic continuation of the function  $F(z, c, b)$  can be found in [22, 23].

Now on the base of formulas (36), (37) we have

$$\langle \tilde{0} | \varphi^2 | \tilde{0} \rangle = \frac{\zeta(1/2)}{4\pi S_D r_H^{D-1}}, \quad \langle \tilde{0} | T_0^0 | \tilde{0} \rangle = \frac{\zeta(-1/2)}{4\pi S_D r_H^{D+1}}. \quad (45)$$

By taking into account that the quantities  $\zeta_{S^{n+1}}(0)$  and  $\zeta_{S^{n+1}}(-1)$  are finite, from formula (39) we see that at  $s = 1/2$  and  $s = -1/2$  the zeta function  $\zeta(s)$  has simple poles with residues  $\zeta_{S^{n+1}}(0)$  and  $\zeta_{S^{n+1}}(-1)/2$ , respectively. Hence, in general, the vacuum expectation values of the field square and the energy density contain the pole and finite contributions. The remained

pole term is a characteristic feature for the zeta function regularization method. Note that for  $n$  even  $\zeta_{S^{n+1}}(-1) = 0$  and the energy density is finite.

The vacuum expectation value of the field square in the boundary-free geometry  $Ri \times S^{D-1}$  is obtained from the Wightman function (27) taking the coincidence limit. Using the relation

$$C_l^{n/2}(1) = \frac{\Gamma(l+n)}{\Gamma(n)l!}, \quad (46)$$

for the corresponding quantity one finds

$$\langle 0_0 | \varphi^2 | 0_0 \rangle = \langle \tilde{0} | \varphi^2 | \tilde{0} \rangle - \frac{r_H^{1-D}}{\pi^2 S_D} \sum_{l=0}^{\infty} D_l \int_0^{\infty} d\omega e^{-\omega\pi} K_{i\omega}^2(\lambda_l \xi), \quad (47)$$

where  $|0_0\rangle$  is the amplitude for the corresponding vacuum state. For large values of  $\xi$ , by using the asymptotic formulas for the MacDonald function for large values of the argument, we can see that the main contribution into the second term on the right of formula (47) comes from the summand  $l = 0$  and we obtain

$$\langle 0_0 | \varphi^2 | 0_0 \rangle = \langle \tilde{0} | \varphi^2 | \tilde{0} \rangle - \frac{r_H^{1-D} e^{-2\lambda_0 \xi}}{\pi^2 S_D \lambda_0 \xi}. \quad (48)$$

In the limit  $\xi \rightarrow 0$  the second term on the right of (47) diverges. For small values  $\xi$  the main contribution comes from large values  $l$  and this term behaves as  $(r_H/\xi)^{D-1}$ . Hence, near the horizon the boundary-free vacuum expectation value of the field square is dominated by the second term on the right of formula (47) and is negative.

Now we turn to the vacuum expectation value of the energy-momentum tensor. The corresponding operator we will take in the form

$$T_{ik} = \partial_i \varphi \partial_k \varphi + \left[ \left( \zeta - \frac{1}{4} \right) g_{ik} \nabla_l \nabla^l - \zeta \nabla_i \nabla_k - \zeta R_{ik} \right] \varphi^2, \quad (49)$$

with the trace relation (34). In (49)  $R_{ik}$  is the Ricci tensor for the bulk geometry and for the metric (1) it has components

$$R_{ik} = 0, \quad i, k = 0, 1; \quad (50)$$

$$R_{ik} = \frac{n}{r_H^2} g_{ik}, \quad i, k = 2, \dots, D. \quad (51)$$

On the base of formula (49) the corresponding vacuum expectation values are presented in the form

$$\langle 0_0 | T_{ik} | 0_0 \rangle = \lim_{x' \rightarrow x} \nabla_i \nabla'_k G_0^+(x, x') + \left[ \left( \zeta - \frac{1}{4} \right) g_{ik} \nabla_l \nabla^l - \zeta \nabla_i \nabla_k - \zeta R_{ik} \right] \langle 0_0 | \varphi^2 | 0_0 \rangle. \quad (52)$$

By using decomposition (27), the vacuum energy-momentum tensor is presented in the form

$$\langle 0_0 | T_{ik} | 0_0 \rangle = \langle \tilde{0} | T_{ik} | \tilde{0} \rangle + \langle T_{ik} \rangle^{(0)}, \quad (53)$$

where the second summand on the right is given by formula (no summation over  $i$ )

$$\langle T_i^k \rangle^{(0)} = -\frac{\delta_i^k r_H^{1-D}}{\pi^2 S_D} \sum_{l=0}^{\infty} D_l \lambda_l^2 \int_0^{\infty} d\omega e^{-\omega\pi} f^{(i)} [K_{i\omega}(\lambda_l \xi)]. \quad (54)$$

In this formula we use the notations

$$f^{(0)}[g(z)] = \left(\frac{1}{2} - 2\zeta\right) \left[ \left(\frac{dg(z)}{dz}\right)^2 + \left(1 - \frac{\omega^2}{z^2}\right) g^2(z) \right] + \frac{\zeta}{z} \frac{d}{dz} g^2(z) + \frac{\omega^2}{z^2} g^2(z), \quad (55)$$

$$f^{(1)}[g(z)] = -\frac{1}{2} \left(\frac{dg(z)}{dz}\right)^2 - \frac{\zeta}{z} \frac{d}{dz} g^2(z) + \frac{1}{2} \left(1 - \frac{\omega^2}{z^2}\right) g^2(z), \quad (56)$$

$$f^{(i)}[g(z)] = \left(\frac{1}{2} - 2\zeta\right) \left[ \left(\frac{dg(z)}{dz}\right)^2 + \left(1 - \frac{\omega^2}{z^2}\right) g^2(z) \right] - \frac{\lambda_l^2 - m^2}{(D-1)\lambda_l^2} g^2(z), \quad (57)$$

with  $i = 2, 3, \dots, D$  and the indices 0 and 1 correspond to the coordinates  $\tau$  and  $\xi$ . Note that for a minimally coupled scalar field the energy density corresponding to (54) is negative for all values  $\xi$ . As in the case of the field square, for large values  $\xi$  the vacuum expectation values (54) are exponentially suppressed by the factor  $e^{-2\lambda_0\xi}$ . For small values  $\xi$  these expectation values behave as  $(r_H/\xi)^{D+1}$  and diverge on the horizon. This type of horizon divergences is also found in the well-investigated example of a bulk  $Ri \times R^{D-1}$  (see references [16]). Here the situation is similar to that which takes place in the Schwarzschild black hole bulk when the quantum field is prepared in the Boulware vacuum [24]. In the black hole case quantities characterizing vacuum polarization display this singular behavior because no physical realization is possible of a static system whose size is arbitrarily close to the gravitational radius. The nature of horizon divergences in Rindler-like spacetimes is similar: no physical realization is possible to a coordinate system for which the lines with a fixed  $\xi$  are arbitrarily close to the Rindler horizon. In this limit the corresponding particles would move at infinitely large acceleration.

## 4 Vacuum expectation values induced by a spherical brane

In this section we consider the vacuum expectation values induced by the presence of a spherical brane. On the base of formula (23) for the Whightman function, the vacuum expectation value for the field square is presented in the form

$$\langle 0|\varphi^2|0\rangle = \langle 0_0|\varphi^2|0_0\rangle + \langle \varphi^2 \rangle^{(b)}, \quad (58)$$

where the second term on the right is induced by the spherical shell:

$$\langle \varphi^2 \rangle^{(b)} = -\frac{r_H^{1-D}}{\pi S_D} \sum_{l=0}^{\infty} D_l \int_0^\infty d\omega \frac{\bar{K}_\omega(\lambda_l a)}{\bar{I}_\omega(\lambda_l a)} I_\omega^2(\lambda_l \xi). \quad (59)$$

This quantity is negative for Dirichlet boundary condition and is positive for Neumann boundary condition. Similar formula can be derived for the vacuum expectation value of the energy-momentum tensor. In accordance with the relation (23) we can write

$$\langle 0|T_i^k|0\rangle = \langle 0_0|T_i^k|0_0\rangle + \langle T_{ik} \rangle^{(b)}, \quad (60)$$

where  $\langle 0_0|T_i^k|0_0\rangle$  is the vacuum expectation value for the situation without the spherical shell and  $\langle T_{ik} \rangle^{(b)}$  is induced by the presence of the sphere. The latter is finite for the points away from the sphere surface and the horizon, and is given by formula (no summation over  $i$ )

$$\langle T_i^k \rangle^{(b)} = -\frac{\delta_i^k r_H^{1-D}}{\pi S_D} \sum_{l=0}^{\infty} D_l \lambda_l^2 \int_0^\infty d\omega \frac{\bar{K}_\omega(\lambda_l a)}{\bar{I}_\omega(\lambda_l a)} F^{(i)}[I_\omega(\lambda_l \xi)], \quad (61)$$

where the expressions for the functions  $F^{(i)}[g(z)]$  are obtained from the corresponding formulas for the functions  $f^{(i)}[g(z)]$  replacing  $\omega \rightarrow i\omega$ . As it has been explained in section 2, the corresponding quantities in the orbifolded braneworld version of the problem are obtained from (59), (61) with an additional coefficient 1/2 and boundary coefficients (26).

Now let us consider various limiting cases of the general formulas for the brane-induced vacuum expectation values. In the limit  $\xi \rightarrow a$  the expectation values for both field square and the energy-momentum tensor diverge. These surface divergences are well known in quantum field theory with boundaries and are investigated for various type of boundary conditions and geometries. For the points near the brane the main contributions come from large values  $l$ . Using the uniform asymptotic expansions for the Bessel modified function, to the leading order one finds

$$\langle \varphi^2 \rangle^{(b)} \approx -\frac{\delta_B \Gamma\left(\frac{D-1}{2}\right)}{(4\pi)^{\frac{D+1}{2}} (a-\xi)^{D-1}}, \quad (62)$$

for the field square and

$$\langle T_0^0 \rangle^{(b)} \approx \langle T_2^2 \rangle^{(b)} \approx \frac{D(\zeta - \zeta_c)\delta_B}{2^D \pi^{\frac{D+1}{2}} (a-\xi)^{D+1}} \Gamma\left(\frac{D+1}{2}\right), \quad (63)$$

for the components of the energy-momentum tensor,  $\zeta_c = (D-1)/4D$  is the curvature coupling parameter for a conformally coupled scalar, and we have introduced the notation

$$\delta_B = \begin{cases} 1, & B = 0 \\ -1, & B \neq 0 \end{cases}. \quad (64)$$

These leading terms are the same as those for a flat brane in the Minkowski bulk. They do not depend on the mass and Robin coefficients and have opposite signs for Dirichlet and non-Dirichlet boundary conditions. The leading term in the asymptotic expansion of the component  $\langle T_1^1 \rangle^{(b)}$  is obtained from (63) by using covariant continuity equation for the tensor  $\langle T_i^k \rangle^{(b)}$ . This term behaves as  $(a-\xi)^{-D}$ .

For large values of the ratio  $a/r_H$  the quantity  $\lambda_l a$  is large and we can replace the Bessel modified functions with this argument by their asymptotics for large values of the argument. This leads to the formulas (no summation over  $i$ )

$$\langle \varphi^2 \rangle^{(b)} \approx -\frac{e^{-2\lambda_0 a}}{S_D r_H^{D-1}} \frac{A - B\lambda_0}{A + B\lambda_0} \int_0^\infty d\omega I_\omega^2(\lambda_0 \xi), \quad (65)$$

$$\langle T_i^k \rangle^{(b)} \approx -\frac{\delta_i^k \lambda_0^2 e^{-2\lambda_0 a}}{S_D r_H^{D-1}} \frac{A - B\lambda_0}{A + B\lambda_0} \int_0^\infty d\omega F^{(i)}[I_\omega(\lambda_0 \xi)], \quad (66)$$

with the exponential suppression of the brane-induced vacuum expectation values.

In the near horizon limit,  $\xi/r_H \ll 1$ , with fixed  $a/r_H$ , the main contributions into the  $\omega$ -integrals come from small values  $\omega$ . Expanding the functions  $I_\omega^2(\lambda_l \xi)$ , to the leading order one finds (no summation over  $i$ )

$$\langle \varphi^2 \rangle^{(b)} \approx -\frac{r_H^{1-D} \mathcal{I}(a)}{2\pi S_D \ln(2r_H/\xi)}, \quad \mathcal{I}(a) = \sum_{l=0}^\infty D_l \frac{\bar{K}_0(\lambda_l a)}{\bar{I}_0(\lambda_l a)} \quad (67)$$

$$\langle T_0^0 \rangle^{(b)} \approx -\langle T_1^1 \rangle^{(b)} \approx -\frac{\zeta r_H^{1-D} \mathcal{I}(a)}{2\pi S_D \xi^2 \ln^2(2r_H/\xi)}, \quad (68)$$

$$\langle T_i^i \rangle^{(b)} \approx \frac{(4\zeta - 1)r_H^{1-D} \mathcal{I}(a)}{4\pi S_D \xi^2 \ln^3(2r_H/\xi)}, \quad i = 2, 3, \dots \quad (69)$$

As we see the brane-induced part in the vacuum expectation value of the field square vanishes at the horizon, whereas the expectation values of the energy-momentum tensor diverge. Recall that near the horizon the boundary free part of the energy-momentum tensor behaves as  $\xi^{-D-1}$  and the vacuum expectation values are dominated by this part. Note that the function  $\mathcal{I}(a)$  is positive for Dirichlet boundary condition and is negative for Neumann boundary condition. In the large mass limit the brane induced vacuum expectation values are exponentially suppressed by the factor  $e^{-2m(a-\xi)}$ .

## 5 Conclusion

In this paper, we investigate the quantum vacuum effects produced by a spherical brane in the  $(D + 1)$ -dimensional bulk  $Ri \times S^{D-1}$ , with  $Ri$  being a two-dimensional Rindler spacetime. The corresponding line element (1) describes the near horizon geometry of a non-extremal black hole spacetime defined by the line element (2). The case of a massive scalar field with general curvature coupling parameter and satisfying the Robin boundary condition on the sphere is considered. To derive formulas for the vacuum expectation values of the square of the field operator and the energy-momentum tensor, we first construct the positive frequency Wightman function. This function is also important in considerations of the response of a particle detector at a given state of motion through the vacuum under consideration [25]. The Wightman function is presented as the sum of the Wightman function for the boundary-free geometry and the term induced by the presence of the spherical brane. For the points away the boundary and horizon the divergences in the coincidence limit are contained in the first term and hence, the renormalization is needed for this term only. In section 3 we have shown that the the Wightman function for the boundary-free  $Ri \times S^{D-1}$  geometry can be presented in the form of a sum of the Wightman function for the boundary-free  $R^2 \times S^{D-1}$  geometry plus a term which is finite in the coincidence limit. As a result for the renormalization of the vacuum expectation values of the field square and the energy-momentum tensor it is sufficient to renormalize the corresponding quantities for the geometry  $R^2 \times S^{D-1}$ . The latter are point-independent and as a renormalization procedure we employ the zeta function regularization method. The corresponding zeta function can be expressed in terms of the zeta function  $\zeta_{S^{n+1}}(z)$  in the geometry  $R \times S^{n+1}$ , the analytic continuation for which is well investigated in literature. Alternatively the zeta function can be expressed in terms of the function (42). The vacuum expectation values of the field square and the energy-momentum tensor are expressed in terms of the zeta function by formulas (45). In general, they contain pole and finite contributions. In the minimal subtraction scheme the pole terms are omitted. As a result the vacuum expectation values of the field square and the energy-momentum tensor for the boundary-free  $Ri \times S^{D-1}$  geometry are determined by formulas (47), (53), (54). On the horizon these expectation values diverge. The leading terms in the near horizon asymptotic expansions behave as  $(r_H/\xi)^{D-1}$  for the field square and as  $(r_H/\xi)^{D+1}$  for the components of the energy-momentum tensor. The vacuum expectation values induced by a spherical brane in the bulk geometry  $Ri \times S^{D-1}$  are investigated in section 4. Near the brane the vacuum expectation values are dominated by the boundary parts and the corresponding components diverge at the brane. For non-conformally coupled scalars the leading terms in the corresponding asymptotic expansions are given by formulas (62), (63) and are the same as those for an infinite plane boundary in the Minkowski bulk. These terms do not depend on the mass and Robin coefficients and have opposite signs for Dirichlet and non-Dirichlet boundary conditions. For large values of the ratio  $a/r_H$  the brane-induced vacuum expectation values are exponentially suppressed by the factor  $\exp[-2(a/r_H)\sqrt{\zeta n(n+1) + m^2 r_H^2}]$ . For the points near the horizon one has  $\xi/r_H \ll 1$  and the brane-induced vacuum expectation value of the field square vanishes as  $\ln^{-1}(2r_H/\xi)$ . Unlike to the field square, the brane-induced parts in the vacuum expectation values of the energy-momentum tensor diverge on the horizon. In this paper we have considered the vacuum expectation values induced by a spherical brane in the region  $0 < \xi < a$ . By the similar way the corresponding quantities for the region  $\xi > a$  may be investigated. It can be seen that in this region the brane-induced quantities can be obtained from those for the region  $0 < \xi < a$  by the replacements  $I_\omega \leftrightharpoons K_\omega$  in formulas (59) and (61). In the corresponding braneworld scenario the geometry is made up by two slices of the region  $0 < \xi < a$  glued together at the brane with a orbifold-type symmetry condition and the ratio  $A/B$  for bulk scalars is related to the brane mass parameter of the field by formula (26). The corresponding formulae for the Wightman function and the vacuum expectation values of

the field square and the energy-momentum tensor are obtained from those given above with an additional coefficient  $1/2$ . They describe the braneworld in the AdS black hole bulk in the limit when the brane is close to the black hole horizon.

Note that in this paper we have considered quantum vacuum effects induced by boundaries in a prescribed background, i.e. the gravitational back-reaction of quantum effects is not taken into account. This back-reaction could have a profound effect on the dynamical evolution of the bulk model. We do not consider this important extension of the theory, but note that the results presented here constitute the starting point for such investigations.

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